What Machines Can Never Learn:
Models and Limitations

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Outline

- What is learning?
- Learning what?
- What is a machine?
- Various models of learning.
What is learning?

Learning From Examples

\[ eg_1 \]
\[ eg_2 \]
\[ \vdots \]
\[ eg_{k-1} \]
\[ eg_k \]
\[ eg_{k+1} \] \hspace{2cm} \text{Machine} \hspace{2cm} \text{Correct desc 1}
\[ \vdots \]
\[ \text{Correct desc 2} \]
\[ \vdots \]
\[ \text{Correct desc } k-1 \]
\[ \vdots \]

Given examples of target concept, Converge to correct description
Learning what?

Functions on natural numbers

- Model *any* discrete domain
- Well-developed theory
- Very general
What is a machine?

Turing Machine

Character | Current | Next
----------|---------|-------
0         | 1       | 0     
q3        | q3      | q3    
R         | R       |       

Or any reasonable programming language
(Gold 67) Learning in the Limit

\[ f(0) \]
\[ f(1) \]
\[ \vdots \]
\[ f(k - 1) \]
\[ f(k) \]
\[ f(k + 1) \]
\[ \vdots \]

\[ \text{Inductive Inference Machine (IIM)} \]

\[ h_0 \]
\[ h_1 \]
\[ \vdots \]
\[ h_{k-1} \]
\[ h_k = f \]
\[ h_k \]
\[ \vdots \]

IIM eventually converges to index for \( f \)
Syntactic (EXplanatory) Learning

\[ f(0) \]
\[ f(1) \]
\[ \vdots \]
\[ f(k-1) \]
\[ f(k) \]
\[ f(k+1) \]
\[ \vdots \]

\[ h_0 \]
\[ h_1 \]
\[ \vdots \]
\[ h_{k-1} \]
\[ h_k = f \]
\[ h_k \]
\[ \vdots \]
Integer coefficient polynomials are \( EX \) learnable.

Algorithm Poly: input \( x \)

If example contradicts last hypothesis
Then interpolate all examples
    and output the result

Example: target concept is \( 2x^3 - 2x^2 + 3 \)

\[
\begin{array}{ccc}
\text{Example} & \text{Hypothesis} \\
\hline \\
 f(0) = 3 & h(x) = 3 \\
 f(1) = 3 & h(x) = 3 \\
 f(2) = 11 & h(x) = 4x^2 - 4x + 3 \\
 f(3) = 39 & h(x) = 2x^3 - 2x^2 + 3 \\
 f(4) = 99 & h(x) = 2x^3 - 2x^2 + 3 \\
 f(5) = 203 & h(x) = 2x^3 - 2x^2 + 3 \\
\end{array}
\]
Point out

- Ignore efficiency

- Example doesn’t return *index*

- Algorithm never stops
Facts about $EX$ Learning

$EX(M)$ is the set of functions which $M$ learns “syntactically”

$EX$ is the class of all $EX(M)$

- (G67) $REC \notin EX$
- If $C$ is effectively enumerable, $C \in EX$
- (BB75) $EX$ not closed under union
THM: $REC \notin EX$

PF: Assume $REC = EX(M)$, Build $f \in REC$ s.t. $f \notin EX(M)$.

Exists $m, n, e_1, e_2$ st $f = \phi_e$, $e \neq e_i$, and

$$
M(f \cdot 1^m) \downarrow e_1 \quad \text{where } \phi_{e_1} = f \cdot 1^\infty \\
M(f \cdot 1^m \cdot 0^n) \downarrow e_2 \quad \text{where } \phi_{e_2} = f \cdot 1^m \cdot 0^\infty
$$

Let $f = f \cdot 1^m 0^n$ and repeat.

CLAIM: $f$ is recursive (search all $m, n$ pairs, watching for mind changes until enough of $f$ is built to compute...
THM: $C$ is effectively enumerable implies $C \in EX$

PF: Let $C = \{\phi_f(0), \phi_f(1), \ldots\}$.

Algorithm for $C$

If current eg inconsistent with last hypothesis
Then Find least $i$ st $\phi_f(i)$ consistent/w egs
   Output program for $\phi_f(i)$

Example: Algorithm for polynomials

If current eg inconsistent with last hypothesis
Then for $i := \infty$ Do
   Find least poly $p$ with
      sum of coefficients and exps $= i$
      consistent with input egs
   If found, output $p$ and leave loop
THM: EX not closed under union
PF: \( EX^0 \) and \( EX^1 \subseteq EX, \ EX^1 \nsubseteq EX \)

CLAIM: \( EX^1 \subseteq EX \)
PF: For each eg sequence, guess where anomaly is and guess correct patch. One pair of guesses will be correct.

CLAIM: \( EX^1 \nsubseteq EX \)
PF: (Later, see “Anomalies” slide)
Semantic (Behaviorally Correct) Learning

\[ f(0) \quad f(1) \quad \vdots \quad f(k-1) \quad f(k) \quad f(k+1) \quad \vdots \]

\[ h_0 \quad h_1 \quad \vdots \quad h_{k-1} \]

For all \( x \), \( f(x) = f_1(x) = f_2(x) = \cdots \)
Facts about $BC$ Learning

$BC(M)$ the set of functions which $M$ learns “semantically”

$BC$ the class of all $BC(M)$

- (B74) $EX \subset BC$

- (G67) $REC \not\in BC$

- (CS83) $BC$ not closed under union
THM: $REC \not\in BC$—same as in EX

THM: $EX \subseteq BC$
PF: Let $C = \{f : \phi_{f(0)} = 1\}$, $C \not\in EX$ (see anomalies)

On input $\sigma$, output program for $\phi_{f(0)}$ with $\sigma$ patched into first $|\sigma|$ values. $BC$ learns input.

THM: $BC$ not closed under union
PF: Let $SD = \{f : \phi_{f(0)} = f\}$ and $FS = \{f : f = 0\text{almost everywhere}\}$.

CLAIM: $SD \in BC$: output first input and halt.

CLAIM: $FS \in BC$: On input $\sigma$, output program for $\sigma0^\infty$. 
CLAIM: \( FS \cup \notin BC \)

PF: idea: use threat of finite support to guarantee mind change or error.

Build \( f \) st \( \phi_{f(0)} = f \) (by recursion theorem) as follows: Fix \( M \). Given \( \sigma \) so far, there is some \( n \) such that either \( M(\sigma 0^n) \downarrow \) or \( \phi_{M(\sigma)} \neq \phi_{M(\sigma 0^n)} \). If not, then \( \phi_{M(\sigma 0^m)} \in FS \) for some \( m \) and so let \( f(x) = \phi_{M(\sigma 0^m)}(x) \) for \( x \leq |\sigma| + n \) and \( f(x) = 1 \) otherwise. In the first case, diagonalize, in the second, force a mind change.

In any case, there is an \( f \) as desired.
Making Mistakes

\[
\begin{align*}
  f(0) & \\
  f(1) & \\
  \vdots & \\
  f(k-1) & \\
  f(k) & \\
  f(k+1) & \\
  \vdots & \\
\end{align*}
\]

\[\text{Inductive Inference Machine (IIM)}\]

\[
\begin{align*}
  h_0 & \\
  h_1 & \\
  \vdots & \\
  h_{k-1} & \\
  h_k & = f \\
  h_k & \\
  \vdots & \\
\end{align*}
\]

\(h_k\) differs from \(f\) on \(\leq a\) values
Facts about Anomalies

$f \equiv^a g$ when $f$ and $g$ differ on at most $a$ values

$EX^a(M)$ are functions which $M$ EX-learns with $\leq a$ anomalies

$EX^a$ the class of all $EX^a(M)$

- (CS83) $EX = EX^0 \subset EX^1 \subset \cdots EX^*$

- (CS83) $BC = BC^0 \subset BC^1 \subset \cdots BC^*$

- (CS83) $REC \in BC^*$
THM: $C = \{ f : \phi_{f(0)} \in EX^1 - EX \}$

PF: $C \in EX^1$ has been shown: patch $\phi_{f(0)}$ with egs so far to get guess.

Fix $M$ and show there is a $f \in C$ st $M$ doesn’t learn $f$. Move anomaly marker. Let $\sigma = e\tau\diamond 0^n$ be $f$ constructed so far (by recursion theorem).

**Diagonalize** If $\phi_{M(\tau)}(\diamond) \downarrow y$, then replace $\diamond$ with a value which causes $f(\diamond) \downarrow \neq \phi_{M(\tau)}$. Move $\diamond$ to end.

**Mind Change** If $M(\sigma) \downarrow \neq M(\sigma\diamond)$ for some value of $\diamond$, then replace $\diamond$ with that value. Move $\diamond$ to end.

**Extend** Let $\sigma = \sigma 0$. 
Replace $\diamond$ with 0.

If the marker moved io, any machine on this sequence will either be wrong, or fail to converge. Else, any machine will err where the marker came to rest.

Properness of BC and EX hierarchies follows by adding more anomalies to class. EG, $C = \{ f : \diamond f(0) \equiv^k \} \in EX^k - EX^{k-1}$. 
**THM:** $REC \in BC^*$

**PF:** To describe the appropriate $M$, describe $\phi_{M(\sigma)}(x) = f(x)$ for any $f \in REC$:

Input $x$

On training sequence $\sigma$
   - Run the first $|\sigma|$ functions on $0 \ldots |\sigma|$ for $x$ steps
   - For each such function $g$ consistent with $\sigma$
     - compute $g(x)$
     - output first converging answer
Changing Your Mind

\[ \begin{align*}
  f(0) \\
  f(1) \\
  \vdots \\
  f(k - 1) \\
  f(k) \\
  f(k + 1) \\
  \vdots 
\end{align*} \]

\[ \begin{align*}
  h_0 \\
  h_1 \\
  \vdots \\
  h_{k-1} \\
  h_k = f \\
  h_k \\
  \vdots 
\end{align*} \]

Has \( k \) mind changes
Facts about Mind Changes

\( EX_m(M) \) are functions \( M \) \( EX \)-learns with at most \( m \) mind changes

\( EX_m \) is class of all \( EX_m(M) \)

\( EX_*(M) \) are functions \( M \) \( EX \)-learns with finite mind changes

- (CS83) \( EX_m \subseteq EX_{m+1} \)

- (CS83) \( BC_m = EX_m \)

- (CS83) \( EX^0_m \subseteq EX^1_m \subseteq \cdots EX^*_m \)
THM: $EX_m \subset EX_{m+1}$
PF: Show $m = 0$. Use functions which equal zero everywhere except at a single accumulation point. If $M(0^n) \downarrow e$, then $M$ misses the function $st \ f(n + 1) = 1$.

THM: $BC_m = EX_m$
PF: Choose an IIM, count mind changes. After $m$, if that happens, stop changing mind. This is an $EX$ algorithm with $\leq m$ mcs.

THM: MC hierarchy proper with fixed anomalies
PF: same proofs carry over

NOTE: this means mind changes are powerful—they conflate syntax and semantics.
Being a Team Player

Team learns $f$ if one machine does
Facts about Team Learning

\[ [m, n]EX^a_b \] Classes of functions EX learned by \( m \) out of \( n \) team members with \( a \) anomalies and \( b \) mind changes

- (JS90) \([m, n]EX \subset [m, n + 1]EX \subset \cdots\)

- (S82) \( m \ EX^a \) machines can simulate \( n \ EX^b \) machine iff \( n \leq m \left(1 + \left\lfloor \frac{b}{a+1} \right\rfloor \right)\)

- (D83) Ditto for BC teams

- (JCN??) Ditto for Popperian machines
More Facts about Team Learning

\[ [m, n]EX_b^a \] Classes of functions EX learned by \( m \) out of \( n \) team members with \( a \) anomalies and \( b \) mind changes

- (DPV91) \([1, 2]EX_0 = [3, 6]EX_0 \subset \cdots\)

- (DPV91) \([2, 4]EX_0 = [4, 8]EX_0 \subset \cdots\)

- (DPV91) \([2, 4]EX_0 - [1, 2]EX_0 \neq \emptyset\)

- (BDV92) if \( \frac{24}{49} < a/b \leq \frac{1}{2} \) then \([a, b]EX_0 = [1, 2]EX_0 \) and \([24, 49]EX_0 - [1, 2]EX_0 \neq \emptyset\)
Accumulation points (as for mind changes) works for teams.

Odd thresholds I don’t understand.
Probabilistic Learning

\[ f(0) \quad f(1) \quad \vdots \quad f(k-1) \quad f(k) \quad f(k+1) \quad \vdots \]

Accepts with probability over coin tosses
Facts about Probabilistic Learning

$EX[p]$ Classes $EX$ learned with probability $p$

- (P85) If $\frac{1}{n+1} < p \leq \frac{1}{n}$ then $EX[p] = [1, n]EX$
Learning Formal Languages

\[
\begin{align*}
&\langle w_0, b_0 \rangle \\
&\langle w_1, b_2 \rangle \\
&\vdots \\
&\langle w_{k-1}, b_{k-1} \rangle \\
&\langle w_k, b_k \rangle \\
&\langle w_{k+1}, b_{k+1} \rangle \\
&\vdots
\end{align*}
\]

\[
\text{gram}_0 \quad \text{gram}_1 \quad \vdots \quad \text{gram}_{k-1} \quad \text{gram}_k \quad \vdots
\]

\text{gram}_k \text{ is grammar for target language } L \text{ and } \quad b_i = L(w_i)
Facts about Language Learning

- (Angluin) Context sensitive grammars can be learned

- But not from positive examples alone
• Other “Classical” Models
  – Query inferencing machines
  – Oracle inferencing machines
  – Ordinal inferencing machines

• Other “Efficient” Models
  – Probably approximately correct (PAC)
  – Membership queries
  – Equivalence queries
Facts About Efficient Models

- (V85) Monomials are PAC learnable
- (V85) CNF is PAC learnable
- DFAs are learnable (exactly) with membership and equivalence queries
Conclusions

- Inductive inferencing can be studied rigorously

- Common learning assumptions can be modelled mathematically

- “Common sense” and the truth are orthogonal

- Computational limitations imply learning limitations