Midterm (Answer Key)
CS 590: Theory of Computation
16 March (Lecture No. 27)

This is a 50 minute, closed book examination—though you may bring a single 8.5x11 inch (or A4) sheet of paper with whatever you want written on it. There are 100 possible points. Write your answers on the exam. 

Please be sure to put your name on every page (at the top).

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**Question 1 (5 points)** Is the following function computable? Why or why not?

\[
f(0) = 17 \,
\]
\[
f(x + 1) = \mu y \left( \sum_{i=0}^{y} i^2 > f(x) \right)
\]

**Answer** Yes, since it is composed by recursion from minimalization and the composition of computable functions.

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**Question 2 (5 points)** Is the set of prime Gödel numbers an index set? Is it recursive? Is it recursively enumerable? Answer all three questions, and justify your answers.

**Answer** It is in fact the set of prime numbers, which is recursive. It is not an index set. If it were, it could not be recursive, by Rice’s Theorem.

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**Question 3 (5 points)** Consider the following function:

\[
f(x) = \begin{cases} 
\phi_x(x) + 1 & \text{if } \phi_x(x) \downarrow \\
0 & \text{otherwise}
\end{cases}
\]

Is \( f \) computable? Why or why not?

**Answer** No. Suppose, to the contrary, that \( f(x) = \phi_x \) for some \( e \). Then if \( \phi_e(e) \downarrow \) then \( f(x) \neq \phi_e(e) \). Otherwise \( \phi_e(e) \uparrow \) but \( f(x) \downarrow 0 \). In either case \( f(x) \neq \phi_e(e) \), contrary to assumption.
Question 4 (10 points) Is it possible to write a URM program to take two Gödel numbers $x$ and $y$ as input and to produce the Gödel number of a program which halts precisely when both $P_x$ and $P_y$ halt? Be sure to justify your answer. (Note: do not provide a program or an algorithm.)

**Answer**  This is asking if there is a computable function $s(x,y)$ such that $W_{s(x,y)} = W_x \cap W_y$. By the smn theorem (and Church's thesis), there is indeed such a function, defined so that

$$\phi_{s(x,y)}(z) = \begin{cases} 1 & \text{if } \phi_x(z) \downarrow \text{ and } \phi_y(z) \downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

Question 5 (10 points) Prove that $P = \{ x : W_x \subseteq \text{Primes} \}$ is not recursive, using Rice's Theorem. (“Primes” is the set of all prime numbers.)

**Answer** This is clearly a non-trivial index set (details of which claim are left to the reader).

Question 6 (10 points) Is $P$ in Question 5 recursively enumerable? Why or why not?

**Answer** No. Suppose $d$ is the Gödel number of a program which diverges everywhere. Then $W_d = \emptyset \subseteq \text{Primes}$, so $d \in P$. By a theorem in class, this makes $P$ productive, and hence not r.e.

Question 7 (10 points) Are there any recursive, creative sets? Why or why not?

**Answer** There are none.

For a set $C$ to be recursive, its complement $\overline{C}$ would have to be r.e. However, to be creative, $\overline{C}$ must be productive, which means there is a computable function $p(x)$ such that if $W_e \subseteq \overline{C}$, then $p(\epsilon) \in \overline{C} - W_e$, and so $\overline{C} = W_e$ is impossible for any $\epsilon$. 
Question 8 (10 points) Show that every set in the $m$ degree of $\overline{K}$ is productive.

**Answer** Let $A$ be in the $m$ degree of $\overline{K}$. Then by definition, $\overline{K} \leq_m A$, so $A$ is productive by the theorem we saw in class.

Question 9 (10 points) If $P$ is a productive set, can $\overline{P}$ be simple? Why or why not?

**Answer** No. A set is simple only when it’s complement has no infinite r.e. subset. However, every productive set has an infinite r.e. subset. So if $\overline{P}$ were simple, then $\overline{\overline{P}} = P$ could have no infinite r.e. subset, and hence could not be productive.

Question 10 (10 points) Prove that if $A \leq_T B$ and $B \leq_T C$ then $A \leq_T C$.

**Answer** Let $a$ and $b$ be such that $\phi^B_a = \chi_A$ and $\phi^C_b = \chi_B$. We know such $a$ and $b$ exist by definition of the two given reductions. Now, let $P^{(1)}_c$ be the following algorithm

Simulate $P_a(x)$
If $O(n)$ is executed (an oracle query)
Then simulate $P_b(r_n)$
   Consult oracle for all queries
   Use answer ($\chi_B(r_n)$) in outer simulation
Return answer from $P_a(x)$

Then clearly $\phi^C_c = \chi_A$, since all queries about $C$ will be answered correctly by access to the oracle, and all queries about $B$ will be answered correctly by simulating $\phi_b$. So, by definition, $A \leq_T C$. 
Question 11 (10 points) Show that $K \leq_m TOT$ (recall that $TOT = \{e : W_e = N\}$).

Answer Define $f(x)$ by the smn theorem so that

$$
\hat{\phi}_{f(x)}(y) = \begin{cases} 
0 & \text{if } x \in K \\
\uparrow & \text{otherwise}
\end{cases}
$$

Now, $x \in K$ iff $\hat{\phi}_{f(x)} = 0$ iff $f(x) \in TOT$, so $f$ is the desired reduction.

Question 12 (5 points) Suppose that $TOT \leq_m A$ and $A = \{f(0), f(1), \ldots\}$. What can you conclude about $f$?

Answer It cannot be a total, recursive function. Otherwise, $A$ would be recursively enumerable, which would in turn imply that $TOT$ is r.e. (by downward closure), which it is not.