Final (Answer Key)
CS 590: Theory of Computation

This is a 110 minute (though you probably won’t need that much time), closed book examination—though you may bring a single 8.5x11 inch (or A4) sheet of paper with whatever you want written on it. There are 100 possible points, in 10 questions on 4 pages. Write your answers on the exam.
Show your work. Partial credit will be given for incorrect answers, provided that the logic behind the answer is clear and shows an understanding of the subject matter.
Please be sure to put your name on every page (at the top).

Question 1 (10 points) Suppose I show that P(A) ≠ NP(A) for a finite set A. What can I conclude about P and NP?

Answer  P ≠ NP, since the finite oracle can be encoded as a finite lookup table and decided in deterministic polynomial time.

Question 2 (10 points) What is the relationship between Mahaney’s theorem (that there is a sparse NP-complete set if and only if P = NP) and the isomorphism conjecture (that all NP-complete sets are p-isomorphic)?

Answer  If the isomorphism conjecture is true, then there are no sparse NP-complete sets, since no sparse set can be isomorphic to SAT, which is very dense. If it is false, then there may be sparse NP-complete sets, but only at the expense of making all sets in NP complete, by having P = NP.

Question 3 (10 points) We know that there are oracles \(A\) and \(B\) such that IP(A) = PSPACE(A) and IP(B) ≠ PSPACE(B). But, Shamir proved that IP = PSPACE without any oracle. What can you conclude about Shamir’s proof from this information alone?

Answer  It did not relativize.
Question 4 (10 points) What is the relationships between $\leq_{\text{log}}$ and $\leq_{m}$?

Answer

Question 5 (10 points) Arrange the following complexity classes according to which is a subset of which: $P, LOG, NPSPACE, EXP = \bigcup \text{TIME}(2^n), NP, PSPACE$.

Answer $LOG \subseteq P \subseteq NP \subseteq PSPACE = NPSPACE \subseteq EXP$

Question 6 (10 points) For which complexity classes are the following problems complete (with respect to $\leq_{m}$ reductions)?

- Satisfiability: Given a boolean formula, is there some assignment of truth values to its variables which makes the formula true?
- Graph Accessibility: Given a directed graph with two distinguished nodes $s$ and $t$, is there a directed path from $s$ to $t$?
- Quantified Boolean Formulae: Given a boolean formula with quantifiers for each variable, is it true?
- Circuit Value: Given a boolean circuit and fixed inputs, is the output 1?

Answer

$NP$ Satisfiability
$NLOG$ Graph Accessibility
$PSPACE$ Quantified Boolean Formulae
$P$ Circuit Value
Question 7 (10 points) Prove that $PSPACE = NPSPACE$

**Answer** $PSPACE \subseteq NPSPACE$, since each deterministic Turing machine can be simulated by a nondeterministic Turing machine which makes no nondeterministic moves.

Let $A$ be a set in $NPSPACE$ via a machine which uses space in $O(n^c)$ for some constant $c$. By Savitch’s theorem, $A \in DSPACE(n^{2c})$, and so $A \in PSPACE$. This shows that $NPSPACE \subseteq PSPACE$.

And so, $PSPACE = NPSPACE$, as was to be shown.

Question 8 (10 points) Prove that $\leq_m^p$ is reflexive and transitive (i.e., for any $A$, $B$, and $C$, $A \leq_m^p A$ and if $A \leq_m^p B$ and $B \leq_m^p C$ then $A \leq_m^p C$).

**Answer** See the book.

Question 9 (10 points) Say a nondeterministic Turing machine accepts a language *unambiguously* if it has exactly one accepting path in its computation tree for every string in the language (and none for those strings not in the language). Let $UP = \{ L : \text{some NTM } M \text{ accepts } L \text{ unambiguously in polynomial time} \}$.

What is the relationship between $P$, $NP$, and $UP$?

**Answer** $P \subseteq UP \subseteq NP$.

Any DTM running in polynomial time is also an NTM in poly time with exactly one path. So $P \subseteq UP$.

Also, if $L \in UP$ then there is a poly time NTM which accepts it, so $UP \subseteq NP$. 
Question 10 (10 points) We know that the probability that \( P = NP \) when relativized to a random oracle is zero. What does this mean? Does this imply that \( P = NP \) in the unrelativized world? Why or why not?

Answer This result, formally stated is: given a characteristic function \( R \), selected uniformly at random from the space of all possible characteristic functions, the probability that \( P(R) = NP(R) \) is zero.

Alternatively, select a set \( R \) by flipping a fair coin for each possible binary string and including the string only if the coin comes up heads. The probability that \( P(R) = NP(R) \) relative to this oracle \( R \) is zero.

However, oracles do exist relative to which \( P \) and \( NP \) are different. So, it remains possible that \( P \neq NP \) in the unrelativized world. That is, the real world could be like choosing a very unusual oracle in the above random process.